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A COMBINATION THEOREM FOR NEGATIVELY CURVED GROUPS

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1. Introduction

The question addressed in this paper is: When does a negatively curved space (always in the sense of Gromov) result from glueing negatively curved spaces? Since the theorem is somewhat technical, in this introduction we give the corollaries and save the statement of the theorem for §2.

Corollary (mapping torus of a free group automorphism). *Let f be an automorphism of the finitely generated free group $\langle a_1, \cdots, a_n\rangle$. Then the mapping torus $M_f = \langle a_1, \cdots, a_n, t | t a_i t^{-1} = f(a_i)$ for $i = 1, \cdots, n\rangle$ is a negatively curved group if and only if f has no nontrivial periodic conjugacy classes.*

The conjugacy problem is solvable for negatively curved groups [6] and so it is solvable for the groups M_f considered above. As far as we know, this is the first proof of that fact. The next corollary is a weaker version of Thurston's fibering theorem [12], but the proof here is new.

Corollary (mapping torus of a pseudoAnosov). *The mapping torus of a pseudoAnosov homeomorphism of a closed surface of genus larger than one is negatively curved.*

The previous corollaries are special cases of the next one. Let f be an automorphism of the negatively curved group G and let $| \ |$ denote the word metric with respect to some finite generating set for G. The automorphism f is *hyperbolic* [5] if there is an integer m and a number $\lambda > 1$ such that, for all g in G, we have $\lambda|g| \leq \max\{|f^m g|, |f^{-m} g|\}$. A pseudoAnosov homeomorphism of a closed surface of genus larger than one induces a hyperbolic automorphism on the level of fundamental groups. Also, an automorphism of a finitely generated free group with no nontrivial periodic conjugacy classes is hyperbolic [1].

Corollary (mapping torus of a hyperbolic automorphism). *The mapping torus of a hyperbolic automorphism is negatively curved.*

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Here is our (weaker) version of the final glueing step in Thurston's proof of his hyperbolization theorem ([11]–[13]). Let F be an orientable incompressible surface in the closed orientable three-manifold M. Let M' be the compact three-manifold resulting from cutting M open along F.

Corollary (final glueing). *Suppose that M' is negatively curved and that the inclusion of each component of $\partial M'$ into M' lifts to a quasi-isometric embedding between universal covers. Then M is negatively curved if and only if M is atoroidal.*

Finally we examine amalgams over virtually cyclic groups.

Corollary (amalgams over virtually cyclics). *Free products and HNN extensions of negatively curved groups over virtually cyclic groups are negatively curved if and only if the resulting groups contain no Baumslag-Solitar groups.*

The main reference for negatively curved groups is Gromov's seminal article [6]. There are also various sets of notes based on this article ([7]–[9]). Cannon's beautiful early investigations [2] must also be mentioned.

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2. Statement of the theorem

Next, we supply the definitions needed to state the theorem.

Graphs of spaces. Let X be a connected finite cell complex with fundamental group G and let $p: X \to \Gamma$ be a map onto a finite graph Γ. Denote the preimage under p of the midpoint of an edge e of Γ by X_e. We require that X_e can be bicollared in X with the collaring respecting the projection to the edge e. Consider the component containing v of Γ cut open along the midpoints of edges. Let X_v denote the preimage under p of this component. We further require that each X_e and X_v be connected and that their inclusions into X induce inclusions of fundamental groups. There is an induced map $\tilde{p}: \widetilde{X} \to T$ from the universal cover of X to a G-tree T such that T/G is isomorphic to Γ. (Throughout the text, the symbol $\sim$ will indicate universal cover.) We call X a *graph of spaces.* The most familiar examples of graphs of spaces are constructed as Eilenberg-Mac Lane spaces associated to graphs of groups [10], however, the universal cover of a graph of spaces need not be contractible.

Negatively curved spaces. Let Z be a finite cell complex. Assigning each edge of Z a length of one induces a combinatorial metric on the one-skeleton of $\widetilde{Z}$ which may be extended to a metric $d_{\widetilde{Z}}$ on $\widetilde{Z}$. There is a similar notion of combinatorial area in $\widetilde{Z}$. The cell complex is *negatively curved* [6] if there is a constant $A = A(Z)$ such that every inessential circuit bounds a disk of combinatorial area less than A times the combinatorial length of the circuit. We denote combinatorial length in Z by l_Z (or just l if the space is understood). A graph of spaces X is called a *graph of negatively curved spaces* if, for each vertex v of Γ, the space X_v is negatively curved. For such a space, set $A(X) = \max\{A(X_v)|v \in \mathrm{Vertex}(\Gamma)\}$. A group is *negatively curved* if it is the fundamental group of a finite negatively curved cell complex.

The qi-embedded condition. The graph of spaces X is said to satisfy the *q(uasi)i(sometrically) embedded condition* if there is a constant $\tau = \tau(X)$ such that $d_{\widetilde{X}_e}(a, b) \leq \tau \cdot \min\{d_{\widetilde{X}_v}(a, b), d_{\widetilde{X}_w}(a, b)\}$ whenever v and w are the endpoints of an edge e in T and a and b are points in $\widetilde{X}_e$. Equivalently, $\widetilde{X}_e$ is quasi-isometrically embedded in $\widetilde{X}_v$ and $\widetilde{X}_w$. (Here $\widetilde{X}_e$ is the subspace of $\widetilde{X}$ associated to the edge e. Note that $\widetilde{X}_e$ is isomorphic to the universal cover of $X_{e/G}$. The analogous statement holds for $\widetilde{X}_v$.) It follows easily from the thin triangles characterization of negatively curved spaces [6] that if X is a graph of negatively curved spaces satisfying the qi-embedded condition, then the edge spaces X_e are also negatively curved.

Hallways and annuli. Hallways and annuli in graphs of spaces will play an important role in what follows. Let m be a positive integer. A disk $\Delta\colon [-m, m] \times I \to \widetilde{X}$ is a *hallway* of *length* $2m$ if it satisfies the following conditions:

(1) $\Delta^{-1}(\bigcup\{\widetilde{X}_e|e \in \mathrm{Edge}(T)\}) = \{-m, -m+1, \cdots, m\} \times I$,

(2) Δ is transverse, relative condition (1), to $\bigcup\{\widetilde{X}_e|e \in \mathrm{Edge}(T)\}$, and

(3) $\Delta|_{(i)\times I}$ is a geodesic in $\widetilde{X}_{e(i)}$, where $\Delta(\{i\} \times I) \subset \widetilde{X}_{e(i)}$ for $i \in \{-m, -m+1, \cdots, m\}$. (Throughout the text, "geodesic" means "length minimizing path.")

Let $\lambda > 1$. The hallway Δ is λ*-hyperbolic* if

$$\lambda l(\Delta(\{0\} \times I)) \leq \max\{l(\Delta(\{-m\} \times I)), l(\Delta(\{m\} \times I))\}.$$

Suppose $\Delta([i, i+1] \times I) \subset \widetilde{X}_{v(i)}$. Then Δ is ρ*-thin* if $d_{\widetilde{X}_{v(i)}}(\Delta((i, t)), \Delta((i+1, t))) \leq \rho$ for $i \in \{-m, -m+1, \cdots, m-1\}$ and $t \in I$. The *girth* of Δ i $l(\Delta(\{0\} \times I))$. The hallway Δ is *essential* if $\Delta|_{[i,i+1]\times\{0\}}$ is not homotopic rel endpoints into $\widetilde{X}_{e(i)}$ for $i \in \{-m, -m+1, \ldots, m-1\}$. Note that the hallway Δ is essential iff the edge path $e(-m)e(-m+1)$, $\cdots, e(m)$ never backtracks in T (i.e., $e(i) \neq e(i+1)$ for $i \in \{-m, -m+1, \cdots, m-1\}$).

A map $\Delta: [-m, m] \times S^1 \to X$ is an *annulus* of length $2m$ in X if the induced map $\widetilde{\Delta}: [-m, m] \times [0, 1] \to \widetilde{X}$ is a hallway of length $2m$. (Here $S^1 = \mathbf{R}/\mathbf{Z}$.) The *girth* of Δ is the length of $\Delta(\{0\} \times S^1)$. The annulus Δ is respectively ρ*-thin* or λ*-hyperbolic* if the induced hallway $\widetilde{\Delta}$ is respectively ρ-thin or λ-hyperbolic. The annulus Δ is essential if $\Delta(\{0\} \times S^1)$ is essential in X and the induced hallway is essential. Notice $\Delta(\{i\} \times S^1)$ is a geodesic in $\widetilde{X}_{e(i)}$ possibly broken at $\Delta(\{i\} \times \{0\})$.

Annuli flare condition. The graph of spaces X is said to satisfy the *annuli flare condition* if there are numbers $\lambda > 1$ and $m \geq 1$ such that for all ρ there is a constant $H(\rho)$ such that any ρ-thin essential annulus of length $2m$ and girth at least H is λ-hyperbolic.

We can now state the theorem.

Theorem (combination theorem). *Let X be a graph of negatively curved spaces. Suppose that X satisfies the qi-embedded and annuli flare conditions. Then X is negatively curved.*

3. More on negatively curved spaces

In a negatively curved space all the geodesic triangles are uniformly thin. More precisely there is a $\delta > 0$ such that the δ-neighborhood of two sides of a geodesic triangle contains the third side. Here is a generalization of the statement that all geodesic triangles in a negatively curved space are δ-thin.

Proposition (Gromov [6, 6.3, Lemma, p. 183, and 72.A] resolution of a quasigeodesic polygon). *Let Z be negatively curved and let $\tau \geq 1$ be a constant. There is a function $B(x) = O(\log x)$ ("big oh" notation) and a linear function $C(x)$ each depending only on Z and τ with the following property. If $\Delta: D^2 \to Z$ is a disk with boundary a k-sided τ-quasigeodesic polygon, then there is a finite* $\mathbf{R}$*-tree S and a map $r: D^2 \to S$ such that:*

(1) *the number of valence one vertices of S is k,*

(2) *for a and b in S^1, $d_Z(\Delta(a), \Delta(b)) \leq d_S(r(a), r(b)) + B(k)$,*

(3) *$r^{-1}(s)$ is a property embedded finite tree in D^2 for $s \in S$,*

(4) *if E is an edge of S, then r restricted to $r^{-1}(\text{Interior}(E))$ is an I-bundle.*

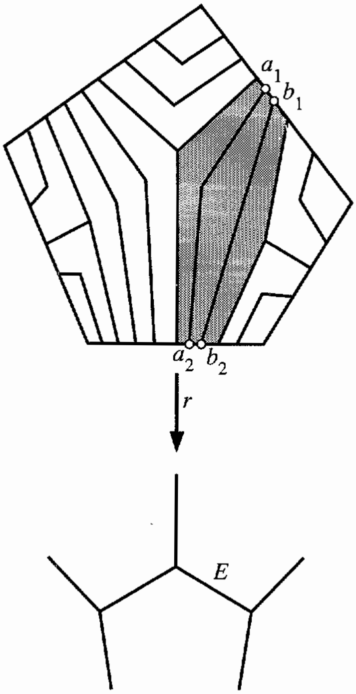

FIGURE 1. SHADED REGION $= r^{-1}(\text{Interior}(E))$.

(5) *for* a_1, b_1 (*respectively* a_2, b_2) *in the same side of the polygon and satisfying* $r(a_1) = r(a_2) \in E$ *and* $r(b_1) = r(b_2) \in E$, *we have*

$$l(\Delta(\textit{the circular arc } a_1b_1 \textit{ in an edge of the polygon})) \\ \leq C(l(\Delta(\textit{circular arc } a_2b_2 \textit{ in an edge of the polygon}))).$$

We call such a map r a *resolution* of the quasigeodesic polygon. A *singular fiber* of a resolution is a fiber that is not isomorphic to I (see Figure 1).

Isoperimetric inequalities. Recall that a finite complex Z is negatively curved if it satisfies a linear isoperimetric inequality. In fact, Z need only satisfy a subquadratic linear inequality.

Theorem (Gromov [6, 2.3.F], [7] subquadratic is enough). *Let* $D: \mathbf{R} \to \mathbf{R}$ *have subquadratic growth. Let* Z *be a finite complex and assume that each inessential circuit* c *in* Z *bounds a disk of area no more than* $D(l(c))$. *Then* Z *is negatively curved.*

Olshanski has also announced a proof.

4. Proof of the weak combination theorem

This section is devoted to a proof of an a priori weaker combination theorem where the annuli flare condition is replaced by a hallways flare condition. In a later section, the two conditions are shown to be equivalent.

Hallways flare condition. The graph of spaces X is said to satisfy the *hallways flare condition* if there are numbers $\lambda > 1$ and $m \geq 1$ such that for all ρ there is a constant $H(\rho)$ such that any ρ-thin essential annulus of length $2m$ and girth at least H is λ-hyperbolic.

Theorem (weak combination theorem). *Let X be a graph of negatively curved spaces. Suppose that X satisfies the qi-embedded and hallways flare conditions. Then X is negatively curved.*

Throughout this section, X is a space satisfying the hypotheses of the weak combination theorem and $A = A(X)$. Also let $B(x) = O(\log x)$ and linear $C(x)$ be functions that satisfy properties (1)–(5) of resolutions simultaneously for each of the vertex spaces of $\widetilde{X}$.

We may assume that the constant λ in the hallways flare condition is any number larger than one by concatenating hallways. We take λ to be four. Let $c: S^1 \to \widetilde{X}$ be a circuit. We may assume c is transverse to and has nonempty intersection with $\bigcup\{\widetilde{X}_e | e \in \mathrm{Edge}(T)\}$.

Good disks. There is a disk $\Delta: D^2 \to \widetilde{X}$ with boundary c that is transverse to $\bigcup\{\widetilde{X}_e | e \in \mathrm{Edge}(T)\}$ and thus dividing D^2 into regions which map into the negatively curved $\widetilde{X}_v$'s. Set $\mathscr{W} = \Delta^{-1}(\bigcup\{\widetilde{X}_e | e \in \mathrm{Edge}(t)\})$.

We may assume Δ has the following properties.

(1) The set $\mathscr{W}$ consists of properly embedded arcs in D^2.

(2) The length of $\Delta(\bigcup\mathscr{W})$ in $\widetilde{X}$ is minimal over all disks satisfying (1).

(3) The closures of the components of $\Delta(D^2 \backslash (\bigcup\mathscr{W}))$ have areas bounded by A times the length of their boundaries.

(4) Define $\mathscr{L}$ to be the set of closures of the components of $S^1 \backslash (S^1 \cap \bigcup\mathscr{W})$. Since a bigon in a negatively curved space bounds a disk of area bounded by a constant times the length of the longer side, we may assume that c restricted to each element of $\mathscr{L}$ is a geodesic in the appropriate $\widetilde{X}_v$. We now view c as a polygon whose sides are elements of $\mathscr{L}$. Notice the number of sides of c is not more than $l(c)$ (see Figure 2).

A disk is *good* if it satisfies (1) through (4).

Resolving good disks. Let $\mathscr{P}$ denote the set of closures of the components of $D^2 \backslash \bigcup\mathscr{W}$. For each $P \in \mathscr{P}$, the map Δ restricted to ∂P is a

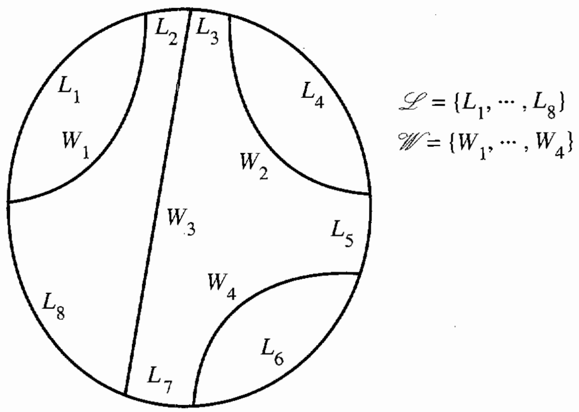

FIGURE 2

τ-quasigeodesic polygon in some $\widetilde{X}_v$ and so we can use Proposition (*resolution of a quasigeodesic polygon*) to resolve P. By glueing the resolutions of the $P \in \mathscr{P}$, we obtain a map $r: D^2 \to S$ where S is a finite tree.

Let $w \in W \in \mathscr{W}$. A *fiber segment* of length i starting at w is an embedding $\sigma: [0, i] \to D^2$ satisfying:

(1) $\sigma(0) = w$,

(2) $\sigma([0, i])$ is contained in a fiber of r, and

(3) $\sigma^{-1}(S^1 \cup (\bigcup \mathscr{W})) = \{0, \cdots, i\}$.

The fiber segment σ is *singular* if $\sigma((0, i))$ meets a singular fiber of the resolution of some $P \in \mathscr{P}$ and *nonsingular* if not. Call w a *singular point* of W if there is a singular fiber segment of length no more than m starting at w. The singular points of W decompose W into a union of closed segments. Denote the set of these segments by $\mathscr{V}(W)$. Set $\mathscr{V} = \bigcup\{\mathscr{V}(W) | W \in \mathscr{W}\}$. For each $V \in \mathscr{V}$, we now have a map $Q_V: [-a_V, b_V] \times I \to D^2$ (pick one of the two orientations) such that for t in the interior of I, the maps Q_V restricted to $[-a_V, 0] \times \{t\}$ and Q_V restricted to $[0, b_V] \times \{t\}$ are nonsingular fiber segments of length at most m. Further, if $a_V < m$, then $Q_V(\{-a_V\} \times I) \subset S^1$ and if $b_V < m$, then $Q_V(\{b_V\} \times I) \subset S^1$. Recall that for the resolutions of the $P \in \mathscr{P}$ the distance between the image under Δ of the endpoints of nonsingular fibers is no more than B(the number of sides of P) $\leq B(l(c))$. (The polygon c has at least as many sides as P and the length of the image under Δ of each of these sides is at least one.) See Figure 3, next page.

It will be important that the size of $\mathscr{V}$ is controlled.

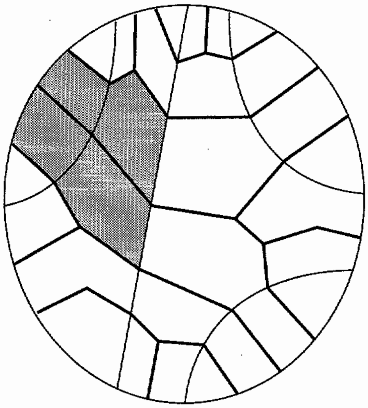

FIGURE 3. SUPPOSE $m = 1$. THE BOLDER LINES INDICATE THE SINGULAR FIBERS OF r. $\circ =$ A SINGULAR POINT OF THE SEGMENT W_i; THESE SINGULAR POINTS BOUND A SEGMENT V. THE SHADED REGION IS $Q_V([-a_V, b_V] \times I)$. HERE $a_V = b_V = 1$.

Lemma (cardinality of $\mathscr{V}$). *Cardinality* $(\mathscr{V}) = O(l(c))$.

Proof. The number of edges in the resolving tree for $P \in \mathscr{P}$ is O(number of sides of P) and so the number of edges in the tree S is O(number of sides of c) $= O(l(c))$. Thus, the number of components of $\{F \setminus \bigcup \text{Vertices}(F) | F$ is a singular fiber of $r\}$ is $O(l(c))$. For $V \in \mathscr{V}$, each component of $Q_V([-a, b] \times \partial I)$ meets an element of $\{F \setminus \bigcup \text{Vertices}(F) | F$ is a singular fiber of $r\}$ or a vertex of c. Further, each element of $\{F \setminus \bigcup \text{Vertices}(F) | F$ is a singular fiber of $r\}$ meets at most $8m$ elements of $\{Q_V([-a_V, b_V] \times \partial I) | V \in \mathscr{V}\}$. The lemma now follows.

Key inequality. For $V \in \mathscr{V}$, define $l_0(V) = l(\Delta Q_V(\{0\} \times I)) = l(\Delta(V))$. Also define $l_+(V) = l(\Delta((\bigcup \mathscr{W}) \cap Q_V(\{b_V\} \times I)))$ and $l_+(V, c) = l(\Delta(S^1 \cap Q_V(\{b_V\} \times I)))$. Similarly define $l_-(V)$ and $l_-(V, c)$.

Lemma (key inequality). *There is a linear function $M(x)$ and a function $N(x) = O(\log x)$ such that if $V \in \mathscr{V}$, then*

$$l_0(V) \leq \tfrac{1}{3}(l_-(V) + l_+(V)) + M(l_-(V, c) + l_+(V, c)) + N(l(c)).$$

Before proving the key inequality, we show how the proof of the weak combination theorem follows from the key inequality.

Proof of the weak combination theorem given the key inequality. By good disk property (3), the area of Δ is bounded by $A \cdot (2l(\Delta(\bigcup \mathscr{W})) + l(\Delta(\bigcup \mathscr{L})))$. We need to bound $l(\Delta(\bigcup \mathscr{W}))$ in terms of $l(\Delta(\bigcup \mathscr{L})) = l(c)$.

Notice that each $w \in \bigcup \mathscr{W}$ which does not lie on a singular fiber of r is in $Q_V(\{-a_V, b_V\} \times I)$ for at most two $V \in \mathscr{V}$. Also note each point of S^1 is in $Q_V(\{-a_V, b_V\} \times I)$ for at most m of the $V \in \mathscr{V}$. Thus,

$$\begin{aligned} l(\Delta(\bigcup \mathscr{W})) &= \sum l_0(V) \\ &\leq \sum [\tfrac{1}{3}(l_-(V) + l_+(V)) + M(l_-(V, c) + l_+(V, c)) + N(l(c))] \\ &\leq \sum \tfrac{2}{3} l_0(V) + M(ml(c)) + N(l(c)) \cdot \text{Cardinality}(\mathscr{V}), \end{aligned}$$

where the sums range over $V \in \mathscr{V}$. We now see that $l(\Delta(\bigcup \mathscr{W})) = O(l(c) \log l(c))$. By Theorem (*subquadratic is enough*), X is negatively curved.

Proof of the key inequality. The key inequality will follow from a series of lemmas. The next lemma, which states that the inequality holds for $V \in \mathscr{V}$ such that $Q_V(\{-a_V, b_V\} \times I)$ meets S^1, follows easily from property (5) of resolutions of quasigeodesic polygons.

Lemma (Q_V's near boundary are controlled). *There is a linear function $M(x)$ such that if $Q_V(\{b_V\} \times I) \subset S^1$, then $l_0(V) \leq M(l_+(V, c))$.*

If $Q_V(\{-a_V, b_V\} \times I)$ does not meet S^1, then $a_V = m = b_V$ and ΔQ_V is a hallway of length $2m$. In order to apply the hallways flare condition, we need to know that some of these hallways are essential.

Lemma (large girth hallways are essential). *Suppose $\Delta Q_V: [-m, m] \times I \to \widetilde{X}$ is a hallway. Suppose also that $l(\Delta Q_V(\{i\} \times I)) > \tau B(l(c))$ for all $i \in \{-m, \cdots, m\}$. Then ΔQ_V is essential.*

Proof. Suppose $e(i) = e(i+1)$ for some $i \in \{-m, -m+1, \cdots, m-1\}$. By property (2) of resolutions of quasigeodesic polygons, we have

$$d_{\widetilde{X}_{v(i)}}(\Delta Q_V((i, t)), \Delta Q_V((i+1, t))) \leq B(l(c)).$$

Thus, $d_{\widetilde{X}_{e(i)}}(\Delta Q_V((i, t)), \Delta Q_V((i+1, t))) \leq \tau B(l(c))$. An obvious surgery then reduces $l(\Delta(\bigcup \mathscr{W}))$, contradicting property (2) of a good disk.

Lemma (large girth hallways are hyperbolic). *Suppose $\Delta Q_V: [-m, m] \times I \to \widetilde{X}$ is a hallway. Then there is a function $N(x) = O(\log x)$ such that if $l_0(V) \geq N(l(c))$, then ΔQ_V is three-hyperbolic.*

Proof. Fix an $i \in \{0, 1, \cdots, m-1\}$ and set $R = \Delta Q_V$ restricted to $[i, i+1] \times I$. Relative to the endpoints, homotope the edges $R([i, i+1] \times \partial I)$ to be geodesics in $\widetilde{X}_{v(i)}$. Now R is a quasigeodesic quadrilateral in $\widetilde{X}_{v(i)}$ and so may be resolved. That is to say, there is a map $r_R: [i, i+1] \times I \to S_R$, where S_R is a finite **R**-tree with exactly four valence one vertices and with r_R satisfying properties (1) through (5) of resolutions (see

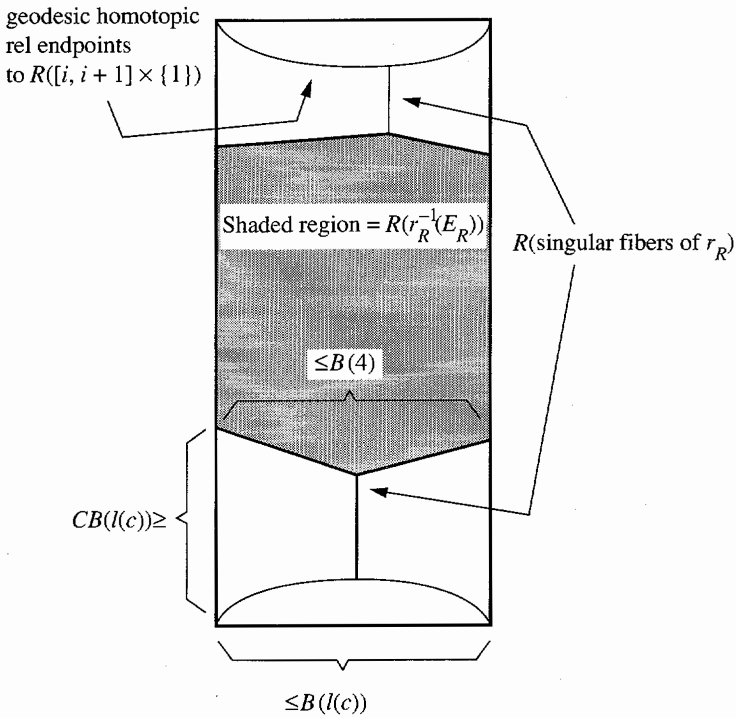

FIGURE 4. $R([i, i+1] \times I) = \Delta Q_V([i, i+1] \times I)$.

Figure 4). Denote the edge of S_R not containing a valence one vertex by E_R. Consider the set defined to be $r_R^{-1}(E_R)$ if the fibers of $r_R^{-1}(E_R)$ run between $\{i\} \times I$ and $\{i+1\} \times I$, and the empty set if not. This set fibers a portion of $[i, i+1] \times I$ and any extension to a fibration of all of $[i, i+1] \times I$ satisfying (1) below also satisfies (2) and (3) below. For the duration of the proof of this lemma, $[i, i+1] \times I$ denotes the new product.

(1) $[i, i+1] \times \partial I$ in the new structure equals $[i, i+1] \times \partial I$ in the old structure.

(2) Let $L(j) = d_{\widetilde{X}_{e(j)}}(\Delta Q_V((j, t)), \Delta Q_V(\{j\} \times \partial I))$ for $j \in \{i, i+1\}$ and suppose $L(i) \geq CB(l(c))$. Then $r_R((i, t)) \in E_R$ and so $\Delta Q_V((i+1, t)) \in \widetilde{X}_{e(i+1)}$ and $d_{\widetilde{X}_{v(i)}}(\Delta Q_V((i, t)), \Delta Q_V((i+1, t))) \leq B(4)$.

(3) If $L(i) \geq \max\{CB(l(c)), 3\tau B(l(c))\}$, then

$$\begin{aligned} L(i+1) &\geq d_{\widetilde{X}_{v(i)}}(\Delta Q_V((i+1, t)), \Delta Q_V(\{i+1\} \times \partial I)) \\ &\geq L(i)/\tau - B(4) - B(l(c)) \geq (1/3\tau)L(i). \end{aligned}$$

Similarly refiber $[j, j+1] \times I$ for all $j \in \{0, 1, \cdots, m-1\}$ and analogously refiber $[j-1, j] \times I$ for all $j \in \{0, -1, \cdots, -m+1\}$ to obtain a new product structure on $[-m, m] \times I$. Set

$$K = (3\tau)^m \max\{CB(l(c)), 3\tau B(l(c))\}.$$

Consider the subsegment $[s_0, t_0]$ of I defined by $t \in [s_0, t_0]$ if and only if $d_{\widetilde{X}_{e(0)}}(\Delta Q_V((0, t)), \Delta Q_V(\{0\} \times \partial I)) \geq K$. By (2) and (3), the hallway ΔQ_V restricted to $[-m, m] \times [s_0, t_0]$ is $B(4)$-thin, and by Lemma (*large girth hallways are essential*) applied to ΔQ_V it is also essential. Thus, ΔQ_V restricted to $[-m, m] \times [s_0, t_0]$ is four-hyperbolic as long as $l(\Delta Q_V(\{0\} \times [s_0, t_0])) \geq H(B(4))$.

Assume $l_0(V) - 2K = l(\Delta Q_V(\{0\} \times [s_0, t_0])) \geq H(B(4))$. By hyperbolicity, we may assume $4l(\Delta Q_V(\{0\} \times [s_0, t_0])) \leq l(\Delta Q_V(\{m\} \times [s_0, t_0]))$, so that

$$\begin{aligned} l(\Delta(\{m\} \times I)) &= l(\Delta(\{m\} \times ([0, s_0] \cup [t_0, 1]))) + l(\Delta Q(\{m\} \times [s_0, t_0])) \\ &\geq 4(l_0(V) - 2K). \end{aligned}$$

If $l_0(V) \geq 8K$, then $l(\Delta(\{m\} \times I)) \geq 3l_0(V)$. Thus, we may take $N(l(c))$ to be $\max\{8K, H(B(4)) + 2K\}$. This completes the proof of Lemma (*large girth hallways are hyperbolic*).

We now see that the functions $M(x)$ from Lemma (Q_V's *near boundary are controlled*) and $N(x)$ from Lemma (*large girth hallways are hyperbolic*) are the functions needed in the key inequality. The proof of the key inequality and therefore also the proof of the weak combination is concluded.

5. Some corollaries

Before showing the equivalence of the flare conditions, we prove the corollaries about hyperbolic automorphisms.

Recall that an automorphism f of a negatively curved group G is said to be *hyperbolic* if there is an integer m and a number $\lambda > 1$ such that, for all g in G, we have $\lambda|g| \leq \max\{|f^m g|, |f^{-m} g|\}$. Since different finite generating sets give rise to quasiisometric metrics, the hyperbolicity of an automorphism is independent of the choice.

Corollary (mapping torus of a hyperbolic automorphism). *Let f be a hyperbolic automorphism of the negatively curved group G. Then the mapping torus M_f of f is negatively curved.*

Proof. Fix a finite presentation for G and let Z be the standard complex associated to it. Choose a metric on $\widetilde{Z}$ as in the subsection on negatively curved spaces. The identification of G with the zero-skeleton $\widetilde{Z}^{(0)}$ of $\widetilde{Z}$ is then an isometry. A group is negatively curved if and only if a finite index subgroup is also. This allows us, by taking powers, to assume that f is hyperbolic for $\lambda = 3$ and $m = 1$.

Let h, respectively k, be a cellular self-map of Z inducing f, respectively f^{-1}, on G. The mapping torus T_h has the structure of a graph of spaces where the graph has one vertex and one edge. The edge space is Z. The vertex space is obtained from the mapping cylinder $C_h = Z \times I \cup Z/(z, 1) \sim hz$ by attaching a collar to (the top) Z. For finite complexes, if $Z_1 \subset Z_2$ induces an isomorphism of fundamental groups, then $\widetilde{Z}_1 \subset \widetilde{Z}_2$ is a quasi-isometry [6]. Thus, the graph of spaces T_h satisfies the qi-embedded condition.

The one-skeleta $\widetilde{C}_h^{(1)}$ and $\widetilde{C}_k^{(1)}$ are equal and may be identified with $\Gamma(G)\times\{0, 1\}\cup\{[(g, 0), (fg, 1)] | g \in G\}$, where $\Gamma(G)$ is the Cayley graph of G. Thus, there are obvious retractions $p_f\colon \widetilde{C}_h^{(1)} \to \Gamma(G) \times \{1\}$ and $p_{f^{-1}}\colon \widetilde{C}_h^{(1)} \to \Gamma(G) \times \{0\}$. If a is an edge path in $\widetilde{C}_h^{(1)}$, then $l([p_f(a)]) \leq |f| l(a)$, where $[p_f(a)]$ is a geodesic in $Z \sim Z \times \{1\}$ that is homotopic rel endpoints to $p_f(a)$ and $|f| = \max\{|fe|, |f^{-1}e|$ for e in the generating set on $G\}$. An analogous inequality holds for $p_{f^{-1}}$.

That M_f satisfies the hallways flare condition with $\lambda = 2$ and $m = 1$ follows easily from the following. Since f is hyperbolic with $\lambda = 3$ and $m = 1$, if c is a geodesic in Z then $3l(c) \leq \max\{l([hc]), l([kc])\}$. Let $aba'b'$ be a loop in $\widetilde{C}_h^{(1)}$ composed of four segments with the segment a geodesic in $Z \times \{0\}$ and the segment a' geodesic in $Z \sim Z \times \{1\}$. Suppose $3l(a) \leq l([ha])$. Then

$$3l(a) \leq l([ha]) = l([p_f a]) \leq l(a' = [p_f a']) + l([p_f b]) + l([p_f b']).$$

So, if $l(b)$ and $l(b')$ are both no more than ρ and if $l(a) \geq 2|f|\rho$, then $2l(a) \leq l(a')$. Similarly, if $3l(a') \leq l([ka'])$, $l(b)$ and $l(b')$ are both no more than ρ, and $l(a') \geq 2|f|\rho$, then $2l(a') \leq l(a)$.

A pseusoAnosov homeomorphism of a closed surface of genus larger than one induces a hyperbolic automorphism of the fundamental group. An automorphism of a finitely generated free group with no nontrivial periodic conjugacy classes is hyperbolic [1] as well. (Such automorphisms abound [4].) Since a finite cell complex is negatively curved if and only if its fundamental group is negatively curved [6], Corollary (*mapping torus of a free group automorphism*) and Corollary (*mapping torus of a pseudoAnosov*) now follow.

6. Equivalence of flare conditions and proof of the combination theorem

In this section we establish the equivalence of the annuli flare and hallways flare conditions. Since X is compact, we obtain:

Lemma (finitely many arcs). *Fix a number. There are in X only finitely many combinatorial arcs of length no more than the fixed number.*

Proposition (annuli flare condition iff hallways flare condition). *The space X satisfies the annuli flare condition if and only if it satisfies the hallways flare condition.*

Proof. The "if" direction of the proposition is clear. The idea for the other direction is to show that a long hallway is nearly a concatenation of annuli. We assume that any $(\rho+2)$-thin essential annulus of length $2m$ and girth at least H is eight-hyperbolic. Choose M large enough so that any collection of M arcs in X of length no more than $2m(\rho+2)$ has two arcs that are the same. This is possible by Lemma (*finitely many arcs*). Let $\Delta\colon [-m, m]\times I \to \widetilde{X}$ be a ρ-thin essential hallway of length $2m$ and girth at least $4HM$. There are $0 \le s_0 < t_0 < s_1 < t_1 < \cdots < s_K < t_K \le 1$ such that:

(1) $\Delta([j, j+1]\times\{s_i\})$ and $\Delta([j, j+1]\times\{t_i\})$ are homotopic rel endpoints for $j \in \{-m, \cdots, m\}$ (here we tacitly use the extra 2 in the $(\rho+2)$-thin condition to enable us to assume $\{\Delta(\{-m, \cdots, m\}\times\{s_i\}), \Delta(\{-m, \cdots, m\}\times\{t_i\}) | 1 \le i \le K\} \subset \text{Vertices}(\widetilde{X}))$,

(2) $l(\Delta(\{0\}\times[s_i, t_i])) \ge H$, and

(3) $\sum\{l(\{0\}\times[0, s_0]), l(\{0\}\times[t_0, s_1]), l(\{0\}\times[t_1, s_2]), \cdots, l(\{0\}\times[t_K, 1])\} \le HM$.

By (1) and (2), Δ restricted to any one of $[-m, m]\times[s_i, t_i]$ represents a $(\rho+2)$-thin essential annulus of girth at least H which we will denote by $A_i\colon [-m, m]\times S^1 \to X$ for $i \in \{1, \cdots, K\}$. Let $\mathscr{I}^+ \subset \{1, \cdots, K\}$ be the set of indices determined by $i \in \mathscr{I}^+$ if and only if $l(A_i(\{m\}\times S^1)) \ge 8l(A_i(\{0\}\times S^1))$. Denote the complement of $\mathscr{I}^+ \subset \{1, \cdots, K\}$ by $\mathscr{I}^-$. We may assume

$$\sum\{l(A_i(\{0\}\times S^1)) | i \in \mathscr{I}^+\} \ge \sum\{l(A_i(\{0\}\times S^1)) | i \in \mathscr{I}^-\}.$$

Then

$$\sum\{l(A_i(\{m\} \times S^1))|i \in \{1, \cdots, K\}\}$$
$$\geq 4\sum\{l(A_i(\{0\} \times S^1))|i \in \{1, \cdots, K\}\}.$$

Now

$$\begin{aligned} l(\Delta(\{m\} \times I)) &\geq \sum\{l(\Delta(\{m\} \times [s_i, t_i]))|i \in \{1, \cdots, K\}\} \\ &= \sum\{l(A_i(\{m\} \times S^1))|i \in \{1, \cdots, K\}\} \\ &\geq 4\sum\{l(A_i(\{0\} \times S^1))|i \in \{1, \cdots, K\}\} \\ &\geq \sum\{4l(\Delta(\{0\} \times [s_i, t_i]))|i \in \{1, \cdots, K\}\} \\ &\geq 4(l(\Delta(\{0\} \times I)) - HM) \geq 3l(\Delta(\{0\} \times I)) \end{aligned}$$

if $l(\Delta(\{0\} \times I)) \geq 4HM$. q.e.d.

Theorem (*combination theorem*) is a consequence of the last proposition and Theorem (*weak combination theorem*).

7. More corollaries

In this section, we collect corollaries of the combination theorem.

Corollary (no long annuli implies hyperbolic). *Let X be a graph of negatively curved spaces. Suppose that X satisfies the qi-embedded condition and that there is an upper bound to the length of essential annuli in X. Then X is negatively curved.*

Three-manifolds. Let F be an oriented incompressible surface in the closed oriented three-manifold M. Using F, the three-manifold M has an obvious graph of spaces description where the graph is a circle with one vertex or a segment with two vertices. Let M' be the compact three-manifold resulting from cutting M open along F. Let $C(M)$ denote the characteristic submanifold of M. The following proposition can be proved using standard three-manifold techniques.

Proposition (no long annuli in M). *Suppose M is atoroidal and $\partial M'$ is not contained in $C(M')$. Then there is a bound to the length of an essential annulus in the graph of spaces M.*

If M is atoroidal and $\partial M'$ is contained in $C(M')$, then M is a mapping torus of a pseudoAnosov homeomorphism. So, Corollary (*mapping torus of a pseudoAnosov*), the previous proposition, and Corollary (*no long annuli implies hyperbolic*) imply Corollary (*final glueing*).

Baumslag-Solitar groups and amalgams. We now look at splittings of negatively curved groups over virtually cyclic subgroups. Virtually cyclic groups are precisely the negatively curved groups that do not contain a free group of rank two [6].

It follows from Gromov [6, Corollary 8.2.C] that if, in a negatively curved group, an element s has infinite order and m and n are nonzero integers, then the equation $ts^m t^{-1} = s^n$ implies that the subgroup generated by s and t is virtually cyclic. In particular, $|m| = |n|$ and we say t *nearly commutes* with a power of s. Therefore, the Baumslag-Solitar group of type m, n, $\mathrm{BS}(m, n) := \langle x, t | ts^m t^{-1} = s^n \rangle$, cannot occur as a subgroup of a negatively curved group [3].

Corollary (free products over virtually cyclics). *Let $G = A *_C B$, where A and B are negatively curved groups and C is virtually cyclic. Then the following three conditions are equivalent*:

(1) *G contains no $\mathbf{Z} \oplus \mathbf{Z}$.*

(2) *For $x \in G$, define $C(x) := \{c \in C | xcx^{-1} \in C\}$. At least one of* (a) *or* (b) *below holds*:

(a) *For all $x \in A \backslash C$, $C(x)$ is finite.*

(b) *For all $x \in B \backslash C$, $C(x)$ is finite.*

(3) *G is negatively curved.*

Proof. Construct a complex for G by assembling $C \times [-1, 1]$ and standard complexes for A and B (with C embedded).

(1) implies (2): Suppose, for $x \in A \backslash C$ and $y \in B \backslash C$, both $C(x)$ and $C(y)$ are infinite. Since $C(x)$ and $C(y)$ are then infinite subgroups of the virtually cyclic group C, there are infinite order elements $c \in C(x)$ and $d \in C(y)$. The equation $xcx^{-1} \in C$ implies x nearly commutes with a power of c. Similarly, y and a power of d nearly commute. Thus, there is a common power z of c and d such that x and y both nearly commute with z. It is easy to see that the subgroup $\langle xyxy, z \rangle$ is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$, a contradiction.

(2) implies (3): Assume $C(x)$ is finite for all $x \in A \backslash C$. We will use the combination theorem to show that G is negatively curved. Let $\rho > 0$ be given. Let S be the set of all elements of $A \backslash C$ with word length in A no more than ρ. Let H be larger than the length in A of any element of the finite set $\bigcup\{C(x) | x \in S\}$. There are no ρ-thin essential annuli of length 2 and girth at least H because the waist curve would be an element of $\bigcup\{C(x) | x \in S\}$ of length in A at least H, a contradiction. Since virtually cyclic subgroups of negatively curved groups are always quasiisometrically embedded [6], the combination theorem now applies.

(3) implies (1): This follows from the comments preceding the statement of this corollary. q.e.d.

For infinite order $x \in A$, $C(x)$ is infinite if and only if some power of x is in C. Thus if G is torsion-free, condition (2) is equivalent to:

(2′) At least one of (a) or (b) below holds:

(a) If some nontrivial power of $x \in A$ is in C, then x is in C.

(b) If some nontrivial power of $x \in B$ is in C, then x is in C.

We now have the following strengthening of a result of Gromov [6, §3.3].

Corollary (torsion-free products over **Z**). *Let $G = A *_{\mathbf{Z}} B$, where A and B are negatively curved torsion-free groups and $\mathbf{Z}$ is a maximal cyclic in at least one of A or B. Then G is negatively curved.*

Similar considerations yield:

Corollary (HNNs over virtually cyclics). *Let $G = A_{\phi}$, where A is negatively curved, and $\phi: C \to C'$ is an amalgamating isomorphism between virtually cyclic subgroups. Then the following conditions are equivalent:*

(1) *G contains no* $\mathrm{BS}(m, n)$.

(2) *The set $\{c \in C | xcx^{-1} \in C'\}$ is finite for all $x \in A$.*

(3) *G is negatively curved.*

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